

Young's functional with Lebesgue-Stieltjes integrals

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Abstract. For non-decreasing real functions f and g , we consider the functional $T(f, g; I, J) = \int_I f(x) dg(x) + \int_J g(x) df(x)$, where I and J are intervals with $J \subseteq I$. In particular case with $I = [a, t]$, $J = [a, s]$, $s \leq t$ and $g(x) = x$, this reduces to the expression in classical Young's inequality. We survey some properties of Lebesgue-Stieltjes integrals and present a new simple proof for change of variables. Further, we formulate a version of Young's inequality with respect to arbitrary positive finite measure on real line including a purely discrete case, and discuss an application related to medians of probability distributions and a summation formula that involves values of a function and its inverse at integers.

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1. INTRODUCTION AND FIRST GENERALIZATION

Let f be a continuous and increasing function defined on an interval $I \subset \mathbb{R}$ and let a, s, t be arbitrary points in I . A version of classical Young's inequality [19] states that

$$(1) \quad Y(f; a, s, t) := \int_a^t f(x) dx + \int_{f(a)}^{f(s)} f^{-1}(x) dx \geq tf(s) - af(a).$$

There has been a considerable amount of literature related to this classical inequality. It seems that the first strict analytic proofs were given in [8] and in [5]. A relation between Young's functional and integration by parts in Riemann-Stieltjes integral was used in [2] and [3] for proving various versions of Young's inequality, and also in [1] to derive several interesting applications. The reverse, in the sense that if (1) holds for some functional $Y(f; a, s, t)$ and for all continuous increasing f , then Y is of the form as in (1), has also been a topic of research. Although in this article we do not address the question of reverse, some interesting contributions like [5], [12] or [13] are worth mentioning. Finally, let us recall that the topic is closely related to convexity, complementary Young's functions and Orlicz spaces.

The first recorded result on upper bounds for Young's functional Y was found by Merkle [14] in 1974; in the setup of (1) it reads

$$(2) \quad Y(f; a, s, t) \leq f(a)(t - a) + a(f(s) - f(a)) + \max((t - a)(f(t) - f(a)), (s - a)(f(s) - f(a))),$$

where $s > a, t > a$. In 2008, Minguzzi [16] discovered a closer upper bound also with $s > a, t > a$:

$$(3) \quad Y(f; a, s, t) \leq tf(t) + sf(s) - af(a) - sf(t)$$

and observed that, given f and an $a \in I$, (2) holds for all $s, t > a$ if and only if (3) does, that is, the statements (2) and (3) are equivalent. Incidentally, the same result was stated by Witkowski in [18, Second proof of Theorem 1] as well as (together with the observation about equivalence) by Cerone [6] in 2009.

Extending the idea of [3], one can observe a more general functional:

$$\begin{aligned}
(4) \quad S(f, g; a, s, t) &:= \int_a^t f(x) \, dg(x) + \int_a^s g(x) \, df(x) \\
&= \int_a^t f(x) \, dg(x) + \int_{f(a)}^{f(s)} g(f^{-1}(y)) \, dy,
\end{aligned}$$

where g is an arbitrary continuous and increasing function and integrals are taken in the Riemann-Stieltjes sense. For $g(x) = x$ this reduces to the classical case (1). The next theorem presents our first generalization of (1) and (3).

1.1 Theorem. *Let f and g be continuous increasing functions defined on an interval $I \subset \mathbb{R}$ and let a, s, t be arbitrary points in I . Then*

$$\begin{aligned}
(5) \quad g(t)f(s) - g(a)f(a) &\leq S(f, g; a, s, t) \\
&\leq g(t)f(t) + g(s)f(s) - g(a)f(a) - g(s)f(t),
\end{aligned}$$

with equality in both sides if and only if $s = t$.

Proof. Using integration by parts, we have that $S(f, g; a, s, s) = f(s)g(s) - f(a)g(a)$ and so, for $s \leq t$,

$$\begin{aligned}
S(f, g; a, s, t) &= S(f, g; a, s, s) + \int_s^t f(x) \, dg(x) \\
&\geq f(s)g(s) - f(a)g(a) + f(s)(g(t) - g(s)) = g(t)f(s) - g(a)f(a),
\end{aligned}$$

which is the first inequality in (5). The equality is possible if and only if

$$\int_s^t (f(x) - f(s)) \, dg(x) = 0,$$

that is, if and only if $s = t$. For $s > t$, the proof is based on the relation

$$S(f, g; a, s, t) = S(f, g; a, t, t) + \int_t^s g(x) \, df(x)$$

and then one proceeds as above.

For the second inequality in (5), we start with the observation that

$$S(f, g; a, s, t) + S(g, f; a, s, t) = f(t)g(t) - f(a)g(a) + f(s)g(s) - f(a)g(a)$$

and so,

$$S(f, g; a, s, t) = f(t)g(t) + f(s)g(s) - 2f(a)g(a) - S(g, f; a, s, t).$$

Applying here the first inequality in (5) with $S(g, f; a, s, t)$ we get the second one. \square

Let us remark that left and right inequality in (5) follow from each other, so the equivalence between (1) and (3) is preserved in the above generalization.

In section 3 we offer an ultimate generalization, for the case when f and g are non-increasing and not necessarily continuous. To achieve this goal, we need the material of Section 2. Examples of applications are postponed to Section 4.

2. LEBESGUE-STIELTJES INTEGRAL: CHANGE OF VARIABLES AND INTEGRATION BY PARTS

In the proof of Theorem 1.1 we used integration by parts formula, and the representation in the second line of (4) is due to change of variables $y = f(x)$. These are extensions of formulas in elementary calculus to Riemann-Stieltjes case, and proofs can be found in the classical text [17]. In this section we deal with Lebesgue-Stieltjes integrals of the form $\int f \, dg$, where f and g are non-decreasing functions that may have jumps. The change of variables in this case is based on the notion of generalized inverse (defined bellow, see (8)) and we were not able to locate a strict proof in the existing literature, except a brief note in [17, p.124]. Here we provide a simple proof (Lemma 2.2) for our case with f and g being non-decreasing; it can be extended to a general f , but this will be addressed elsewhere. The second part of this section is devoted to integration by parts in Lebesgue-Stieltjes integrals.

For a non-decreasing function $g : \mathbb{R} \mapsto \mathbb{R}$ one can define a positive measure μ_g on the family of intervals $[a, b]$ by

$$(6) \quad \mu_g([a, b]) := g(b_+) - g(a_-), \quad (a \leq b)$$

and extend it to the Borel sigma field on real line. The Lebesgue-Stieltjes (L-S) integral of a given measurable function f with respect to g is defined as the Lebesgue integral of f with respect to μ_g

$$\int_{\mathbb{R}} f(x) dg(x) := \int_{\mathbb{R}} f(x) d\mu_g(x)$$

If this integral is finite, we say that f is g -integrable. Let us assume that f is g -integrable and non-negative. Then from the theory of Lebesgue integration it is enough to consider a partition of interval $[0, n]$ on the y -axis with subintervals $J_{n,k} = [\frac{k-1}{2^n}, \frac{k}{2^n}]$, $k = 1, 2, \dots, n2^n$, and points $y_{n,k} = \frac{k-1}{2^n}$. Then

$$(7) \quad \int_{\mathbb{R}} f(x) dg(x) = \lim_{n \rightarrow +\infty} \sum_{k=1}^{n2^n} y_{n,k} \mu_g(f^{-1}(J_{n,k})).$$

In a general case, we use the decomposition $f = f_+ - f_-$, where $f_+(x) = \max\{f(x), 0\}$ and $f_-(x) = \max\{-f(x), 0\}$.

The equality (6) can be related to Lebesgue measure on real line, since $\mu_g([a, b]) = g(b_+) - g(a_-) = \text{Leb}([g(a_-), g(b_+)])$. More generally, for an interval I define the interval I_g to be $[g(a_-), g(b_+)]$, $[g(a_-), g(b_-)]$, $[g(a_+), g(b_+)]$, $[g(a_+), g(b_-)]$, if I is $[a, b]$, $[a, b)$, $(a, b]$ or (a, b) , respectively. Sometimes it is convenient to think about empty set as a special interval; for $I = \emptyset$ we define $I_g = [a, a]$ where a is any real number, say zero; then $\mu_g(\emptyset) = \text{Leb}([a, a]) = 0$. In what follows we will need a uniform notation for ends of the closed interval I_g , so let us define $g_*(I)$ and $g^*(I)$ to be left and right endpoint of the closed interval I_g . Note that for a non-decreasing g , $g_*(I) \leq g(x) \leq g^*(I)$ for any $x \in I$. Then on an arbitrary interval I we may define

$$\int_I f(x) dg(x) = \int_{\mathbb{R}} f(x) dg_I(x),$$

where $g_I(x)$ is defined to be $g(x)$ if $g_*(I) \leq g(x) \leq g^*(I)$, and $g_*(I)$ or $g^*(I)$ if $g(x) < g_*(I)$ or $g(x) > g^*(I)$, respectively. If a non-decreasing f is defined only on I , we may extend it to \mathbb{R} in a way analogous to previous case, and so integrals of the form $\int_I f(x) dg(x)$ where f and g are non-decreasing functions can be always expressed via integrals of the form $\int_{\mathbb{R}} f(x) dg(x)$ with some other non-decreasing functions f and g defined on \mathbb{R} . Therefore, we may always assume that the functions f and g are defined on \mathbb{R} .

Now, the general formula analogous to (6) reads

$$\mu_g(I) = \text{Leb}(I_g), \quad \text{i.e.} \quad \int_I dg(x) = \int_{I_g} dy,$$

which is in fact a change of variables $y = g(x)$.

Let g be a non-decreasing (not necessarily continuous) function on I . Following [7] and [3], we say that x a generalized inverse of g at y , $x = g^{-1}(y)$, if

$$(8) \quad \sup\{t \mid g(t) < y\} \leq x \leq \inf\{t \mid g(t) > y\}.$$

The generalized inverse $g^{-1}(y)$ is not unique at a point y if and only if $g(I) = \{y\}$, where I is an interval with non-empty interior; in that case, any rule that will assign an $x \in \bar{I}$ yields a version of $g^{-1}(y)$. From (8) the following properties follow easily:

$$(9) \quad g(x) < y \implies x \leq g^{-1}(y), \quad g(x) > y \implies x \geq g^{-1}(y);$$

$$(10) \quad g(x_-) < y < g(x_+) \implies g^{-1}(y) = x, \quad g^{-1}(y) = x \implies g(x_-) \leq y \leq g(x_+);$$

$$(11) \quad g^{-1}(y) < x \implies y \leq g(x_-), \quad g^{-1}(y) > x \implies y \geq g(x_+);$$

$$(12) \quad g^{-1}(y) \leq x \implies y \leq g(x_+), \quad g^{-1}(y) \geq x \implies y \geq g(x_-).$$

The next lemma gives another way of expressing $\mu_g(I)$, for an interval I .

2.1 Lemma. *Let g be a non-decreasing functions on \mathbb{R} and let I be an interval. For any version of the generalized inverse g^{-1} it holds that $\{y \mid g^{-1}(y) \in I\} \subseteq I_g$ and*

$$(13) \quad \mu_g(I) = \text{Leb}(I_g) = \text{Leb}(\{y \mid g^{-1}(y) \in I\}).$$

Proof. We have to prove the second equality in (13) only. For a given I and a fixed version of g^{-1} , let $J := \{y \mid g^{-1}(y) \in I\}$.

For a singleton $I = \{a\}$, we have that $I_g = [g(a_-), g(a_+)]$ and by the second property in (10) we have that $J \subseteq I_g$. The first property in (10) implies that $I_g \subseteq J \cup \{g(a_-), g(a_+)\}$, hence $\text{Leb}(I_g) = \text{Leb}(J)$.

If I is an open interval $I = (a, b)$ where $a < b$, then $I_g = [g(a_+), g(b_-)]$, and the property (11) shows that $J \subseteq I_g$, whereas the reverse of (12) implies that $I_g \subseteq J \cup \{g(a_+), g(b_-)\}$, and again $\text{Leb}(I_g) = \text{Leb}(J)$.

For other kinds of intervals, the statement follows from additivity. \square

Now we can give the announced proof of change of variables formula in case when f is also non-decreasing.

2.2 Lemma. *Let f and g be non-decreasing functions on an interval I and let g^{-1} be any version of generalized inverse. Then*

$$(14) \quad \int_I f(x) dg(x) = \int_{I_g} f(g^{-1}(y)) dy.$$

Proof. Since f is non-decreasing, then $f^{-1}(J)$ is an interval (or empty set) for any interval J . Then we use Lemma 2.1 to show that terms of summation on the right hand side of (7) are equal for both integrals in (14); therefore, the integrals are equal. \square

The following result on integration by parts for in L-S integrals is due to E. Hewitt [11] (see also [4, Theorem 6.2.2.]).

2.3 Lemma. *Let f and g be real valued functions of finite variation defined on \mathbb{R} and let I be an interval. If $D(I)$ is (at most countable) set of points $d \in I$ where both f and g are discontinuous, then*

$$(15) \quad \int_I f(x) dg(x) + \int_I g(x) df(x) = \mu_{fg}(I) + \sum_{d \in D(I)} A(d),$$

where

$$(16) \quad A(d) = f(d)(g(d_+) - g(d_-)) + g(d)(f(d_+) - f(d_-)) - f(d_+)g(d_+) + f(d_-)g(d_-).$$

Since L-S integral in explicit notations depends on order of points and type of intervals, the explicit form of (15) is different for different types of intervals. For example, if $I = [a, b]$, $a < b$, then

$$(17) \quad \int_{[a,b]} f(x) dg(x) + \int_{[a,b]} g(x) df(x) = f(b_+)g(b_+) - f(a_-)g(a_-) + \sum_{d \in D([a,b])} A(d)$$

In a special case when one of functions f, g is right continuous at d and both are non-decreasing (or both non-increasing)

$$(18) \quad A(d) = (f(d) - f(d_-))(g(d) - g(d_-)) \geq 0.$$

Let us finally remark that, given a finite measure μ , we may define a right continuous function $g_r(x) = \mu(-\infty, x]$ so that $\mu = \mu_{g_r}$ in the sense of L-S measure. Hence for an arbitrary positive finite measure μ and a non-decreasing function f , we may take $g = g_r$ to make summands $A(d)$ non-negative.

3. YOUNG'S FUNCTIONAL-LOWER AND UPPER BOUNDS

In this section we assume that all functions are defined on \mathbb{R} , as explained in the previous section. We also allow a possibility that intervals of integration can be infinite, as far as the value of the integral is finite. Theorem 3.1 below offers one possible version of Young's inequality with Lebesgue-Stieltjes integrals. Here we will use the following notations,

$$\begin{aligned}\bar{f}(I) &= \inf\{f(x) \mid x > I\}, & \underline{f}(I) &= \inf\{f(x) \mid x > I\}, \\ f_{\min}(I) &= \inf\{f(x) \mid x \in I\}, & f_{\max}(I) &= \sup\{f(x) \mid x \in I\},\end{aligned}$$

where I is an arbitrary nonempty interval. We note that for any non-decreasing function f we have

$$\underline{f}(I) \leq f_*(I) \leq f_{\min}(I) \leq f_{\max}(I) \leq f^*(I) \leq \bar{f}(I).$$

In this section we present lower and upper bounds for the functional

$$\begin{aligned}(19) \quad T(f, g; I, J) &:= \int_I f(x) \, dg(x) + \int_J g(x) \, df(x) \\ &= \int_I f(x) \, dg(x) + \int_{J_f} g(f^{-1}(u)) \, du\end{aligned}$$

where I and J are arbitrary intervals with $J \subseteq I$. For $I = J$, the functional (19) can be expressed via integration by parts formula (15). A particular case of (19) that corresponds to classical Young's inequality is with $I = [a, t]$, $J = [a, s]$, $a < s \leq t$ and with $g(x) = x$.

3.1 Theorem. *Let f and g be non-decreasing functions on \mathbb{R} and let I, J be intervals such that $J \subseteq I$. Then*

$$(20) \quad \begin{aligned}T(f, g; I, J) &\geq T(f, g; J, J) + f_{\min}(I)(g_*(J) - g_*(I)) \\ &\quad + \bar{f}(J)(g^*(I) - g^*(J)),\end{aligned}$$

with equality if $I = J$.

In particular, with $a < s \leq t$, we have that

$$\begin{aligned}(21) \quad &\int_{[a, t]} f(x) \, dg(x) + \int_{[a, s]} g(x) \, df(x) \\ &\geq f(s_+)g(t_+) - f(a_-)g(a_-) + \sum_{d \in D([a, s])} A(d),\end{aligned}$$

where $A(d)$ is defined as in (16).

Proof. Let $J \subseteq I$. The set $I \setminus J$ is of the form $A \cup B$, where A and B are disjoint intervals or empty sets with $a < x < b$ for any $a \in A, x \in J, b \in B$. We start from

$$(22) \quad T(f, g; I, J) = T(f, g; J, J) + \int_{A \cup B} f(u) \, dg(u)$$

and use a bound

$$\begin{aligned}(23) \quad \int_{A \cup B} f(u) \, dg(u) &\geq f_{\min}(I)\mu_g(A) + \bar{f}(J)\mu_g(B) \\ &= f_{\min}(I)(g_*(J) - g_*(I)) + \bar{f}(J)(g^*(I) - g^*(J)),\end{aligned}$$

where the second line follows from noticing that $g^*(A) = g_*(J)$, $g_*(A) = g_*(I)$ and analogously for B . Then (20) follows from (22) and (23).

The special case (21) follows from (20) for $I = [a, t]$, $J = [b, s]$, by using the integration by parts formula of Lemma 2.3 for $T(f, g; J, J)$.

3.2 Remark. A simple observation that (23) can be re-phrased in terms of upper bounds, yields an immediate upper bound in Young's inequality. The bound obtained in such a way agrees with the bound that is provided in the next theorem if f and g are continuous functions.

3.3 Theorem. *Under same conditions and notations as in Theorem 3.1,*

$$(24) \quad \begin{aligned} T(f, g; I, J) &\leq T(f, g; I, I) - g_{\min}(I)(f_*(J) - f_*(I)) \\ &\quad - \bar{g}(J)(f^*(I) - f^*(J)), \end{aligned}$$

with equality if $I = J$. In particular, for $a \leq s \leq t$,

$$(25) \quad \begin{aligned} &\int_{[a,t]} f(x) dg(x) + \int_{[a,s]} g(x) df(x) \\ &\leq f(t_+)g(t_+) + f(s_+)g(s_+) - f(a_-)g(a_-) - f(t_+)g(s_+) + \sum_{d \in D([a,t])} A(d) \end{aligned}$$

Proof. Note first that

$$T(f, g; I, J) + T(g, f; I, J) = T(f, g; I, I) + T(f, g; J, J).$$

By Theorem 3.1, we have that

$$\begin{aligned} T(g, f; I, J) &\geq T(g, f; J, J) + g_{\min}(I)(f_*(J) - f_*(I)) \\ &\quad + \bar{g}(J)(f^*(I) - f^*(J)), \end{aligned}$$

which together with the observation that $T(f, g; J, J) = T(g, f; J, J)$ ends the proof.

3.4 Remark. The prescribed order relation $J \subseteq I$, or, in our particular case, $a \leq s \leq t$, in theorems 3.1 and 3.3 is important because the L-S integral is sensitive to the order of limits of integration. However, analogous results can be stated for any other arrangement of a, s, t .

3.5 Remark. It is difficult to state universal necessary conditions for equality in theorems 3.1 and 3.3. As an illustration, consider a function $g(x) = \text{sgn}(x - c)$, for a fixed $c \in \mathbb{R}$, and let f be a continuous non-decreasing function. Then, for $a < c < s < t$, we have that

$$S(f, g; s, t) = \int_{[a,t]} f(x) dg(x) + \int_{[a,s]} g(x) df(x) = f(s) + f(a),$$

which gives equality with both lower and the upper bound. If $a < s < c < t$, then $S(f, g; s, t) = 2f(c) + f(a) - f(s)$, the lower bound is $f(s) + f(a)$ and the upper bound is $2f(t) + f(a) - f(s)$ and equality with the lower bound occurs iff $f(c) = f(s)$, while the equality with the upper bound occurs iff $f(s) = f(t)$.

4. EXAMPLES

4.1 Young's inequalities with respect to a measure. Let μ be a positive finite countably additive measure on Borel subsets of \mathbb{R} . Along the lines of Section 2, we define a right continuous function $G(x) = \mu\{(-\infty, x]\}$. For a non-decreasing function f we have that

$$T(f, G; I, J) = \int_I f(x) d\mu(x) + \int_J G(x) df(x)$$

Then we may observe Young's inequalities as in Section 3. In order to simplify notations, let us assume that μ is the probability distribution of a random variable Y on some abstract probability space (Ω, \mathcal{F}, P) ; then G is the cumulative distribution function of Y , and taking $I = (-\infty, t]$, $J = (-\infty, s]$, we have that

$$(26) \quad \begin{aligned} &f(s_+)G(t) + \sum_{d \in D((-\infty, s])} (f(d) - f(d_-))P(Y = d) \\ &\leq \mathbb{E}(f(Y) \cdot I_{\{Y \leq t\}}) + \int_{[f(-\infty), f(s_+)]} G(f^{-1}(u)) du \\ &\leq f(t_+)(G(t) - G(s)) + f(s_+)G(s) + \sum_{d \in D((-\infty, t])} (f(d) - f(d_-))P(Y = d). \end{aligned}$$

This holds for a general probability measure μ . In a purely discrete case μ is concentrated on a countable set of points $\{y_k\}$, where we may think of $\{y_k\}$ as being an increasing sequence in $k \in (-\infty, +\infty)$ or

$k \in (1, +\infty)$, with corresponding probabilities p_k . In this case the Young's functional with $I = (-\infty, y_n]$ and $J = (-\infty, y_m]$, where $m \leq n$, is expressed by sums as follows.

$$T(f, G; I, J) = \sum_{k=1}^n f(y_k) p_k + \sum_{k=1}^{m-1} G(y_k) (f(y_{k+1-}) - f(y_{k-})) + G(y_m) (f(y_{m+}) - f(y_{m-})).$$

Now suppose that we have another probability measure that corresponds to a random variable X , with a probability distribution function F . Rewriting (26) in this setup, f being replaced with F , we get

$$\begin{aligned} F(s)G(t) + \sum_{d \in D((-\infty, s])} P(X = d)P(Y = d) \\ \leq E(F(Y) \cdot I_{\{Y \leq t\}}) + E(G(X) \cdot I_{\{X \leq s\}}) \\ \leq F(t)(G(t) - G(s)) + F(s)G(s) + \sum_{d \in D((-\infty, t])} P(X = d)P(Y = d). \end{aligned}$$

The case $X = Y$, $F = G$, $s = t = +\infty$ gives a simple formula

$$(27) \quad E F(X) = \frac{1}{2} + \frac{1}{2} \sum_{d \in D} P^2(X = d),$$

where D is the set of atoms of probability distribution for X . Proceeding further, let F^{-1} be any version of inverse; then (27) implies that

$$F^{-1}(E F(X)) \geq F^{-1}\left(\frac{1}{2}\right) \in \text{Med } F,$$

where $\text{Med } F$ denotes the median set (closed interval or a singleton), which is defined as the set of all possible values of $F^{-1}(1/2)$.

By (8), $F^{-1} \leq \bar{F}^{-1}$, where

$$\bar{F}^{-1}(y) = \inf\{t \mid F(t) > y\}$$

and so, if m is any median of X , then

$$m \leq \bar{F}^{-1}(E F(X)).$$

This is a new general relation between median and expectation, that seems not to have been recorded in the literature (see for example [9] or [15] for some results and facts about one dimensional medians).

4.2 A summation formula. In this part, the letters i, j, k, m, n will be reserved for integers only. The following result was proved in [10]: If f is a strictly increasing positive function on $[0, +\infty)$, $0 < f(1) \leq 1$, then for any positive integer n it holds that

$$(28) \quad \sum_{j=1}^n [f(j)] + \sum_{k=1}^{[f(n)]} [f^{-1}(k)] = n[f(n)] + K(1, n),$$

where K is the number of integer points $j \in [1, n]$ such that $f(j)$ is an integer. A generalized version of this result can be derived using the Lebesgue-Stieltjes integral as follows.

We start with (17) with $g(x) = [x]$ and replacing f with $[f]$ (where $[x]$ is the greatest integer $\leq x$). To be consistent with usual notations, we may use $[f](x)$ for $[f(x)]$; note that $[f](x_{\pm})$ and $[f(x_{\pm})]$ need not be equal. For simplicity take $a = m$ and $b = n$ where m and n are integers. Then we have,

$$\begin{aligned} \int_{[m, n]} [f](x) d[x] + \int_{[m, n]} [x] d[f](x) = \\ n \cdot [f](n_+) - (m-1) \cdot [f](m_-) + K(m, n), \end{aligned}$$

where by (18),

$$K(m, n) = \sum_{j=m}^n ([f](j) - [f](j_-)).$$

Now let us fix the version of inverse to be the smallest one:

$$f^{-1}(y) = \sup\{t \mid f(t) < y\}$$

and also assume that f is right continuous. With these assumptions, we have that $J(x) := [f](x_+) - [f](x_-) = [f](x) - [f](x_-) \neq 0$ if and only if $f(x)$ is an integer, and then $J(x)$ equals the number of integers $k \in ([f](x_-), [f](x_+))$, or, in another way, the number of integers k with the property that $f^{-1}(k) = x$. Then

$$\int_{[m,n]} [x] d[f](x) = \sum_{x \in D([m,n])} [x]([f](x_+) - [f](x_-)) = \sum_{k \in ([f](m_-), [f](n_+))} [f^{-1}(k)]$$

and as an immediate consequence we have that

$$(29) \quad \begin{aligned} \sum_{j=m}^n [f(j)] + \sum_{k=[f](m_-)+1}^{[f](n_+)} [f^{-1}(k)] \\ = n \cdot [f](n_+) - (m-1) \cdot [f](m_-) + K(m, n), \end{aligned}$$

where $K(m, n)$ is the number of integers $k \in ([f](m_-), [f](n_+))$ with the property that $f^{-1}(k)$ is also an integer. This is a generalization of (28) for right continuous non-decreasing functions f . The formula could be useful in number theory, for example, to find a numerical value of $K(m, n)$ for some functions f .

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REFERENCES

- [1] R. P. Boas, Jr., M. B. Marcus, *Inequalities involving a function and its inverse*, SIAM J. Math. Anal. **4** (1973), 585–591.
- [2] R. P. Boas, Jr., M. B. Marcus, *Inverse functions and integration by parts*, Amer. Math. Monthly **81** (1974), 760–761.
- [3] R. P. Boas, Jr., M. B. Marcus, *Generalizations of Young's inequality*, J. Math. Anal. Appl. **46**(1974), 36–40.
- [4] M. Carter, B. van Brunt, "The Lebesgue-Stieltjes Integral: A Practical Introduction", Springer 2000.
- [5] P. S. Bullen, *The inequality of Young*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 357–380 (1971), 51–54.
- [6] P. Cerone, *On Youngs inequality and its reverse for bounding the Lorenz curve and Gini mean*, J. Math. Ineq., **3** (2009), 369–381.
- [7] F. Cunningham, Jr, N. Grossman: *On Young's inequality*. Amer. Math. Monthly **78** (1971), 781–783.
- [8] J. B. Diaz, F. T. Metcalf, *An analytic proof of Young's inequality*, Amer. Math. Monthly **77** (1970) 603–609.
- [9] Y. N. Ghosh, B. Mukherjee, *On probabilistic properties of conditional medians and quantiles*, Statist. Probab. Lett. **76** (2006), 1775–1780
- [10] H. Gupta, *Use of inverse functions in summation*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 544-576 (1976), 101-102.
- [11] E. Hewitt, *Integration by parts for Stieltjes integrals*, Amer. Math. Monthly **67** (1960), 419–423.
- [12] Ih-Ching Hsu: *On a converse of Young's inequality*. Proc. Amer. Math. Soc. **33** (1972), 107–108.
- [13] I. B. Lacković, *A note on a converse of Young's inequality*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 461-497 (1974), 73–76.
- [14] M. Merkle, *A Contribution to Young's inequality*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 461-497 (1974), 265–267.
- [15] M. Merkle, *Jensen's inequality for medians*, Statistics & Probability Letters, **71** (2005), 277–281.
- [16] E. Minguzzi, *An equivalent form of Youngs inequality with upper bound*, Appl. Anal. Discrete Math. **2** (2008), 213-216.
- [17] F. Riesz, B. Sz. Nagy, "Functional Analysis", Dover Publications, 1990.
- [18] A. Witkowski, *On Young's inequality*, JIPAM. J. Inequal. Pure Appl. Math. **7** (2006), Article 164.
- [19] W. H. Young, *On classes of summable functions and their Fourier series*, Proc. Royal Soc., Series (A), **87** (1912), 225–229.

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